Linear approximation

Linear approximation is a useful tool because it allows us to estimate values on a curved graph (difficult to calculate), using values on a line (easy to calculate) that happens to be close by.

If we want to calculate the value of the curved graph at a particular point, but we don't know the equation of the curved graph, we can draw a line that's tangent to the curved graph at the point we're interested in. Remember that "tangent to the graph" means that the line barely skims the graph and touches it at only one point.

As you pick values on the line that are closer and closer to the point of tangency, you'll get a better and better approximation of the value of the point, without ever using the equation of the curved graph. When calculating the value of the function at the point of tangency is particularly difficult, it can be convenient to use the linear approximation formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Example

Use linear approximation to estimate $f(1.9)$ by first evaluating $f(x)$ at $a = 2$.

$$f(x) = \frac{2}{\sqrt{x - 1}}$$

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Since we're asked to estimate $f(1.9)$, we can plug $a = 2$ into $f(x)$ to get an estimate that's close to 1.9.

$$f(x) = \frac{2}{\sqrt{x-1}}$$

$$f(2) = \frac{2}{\sqrt{2-1}}$$

$$f(2) = \frac{2}{\sqrt{1}}$$

$$f(2) = 2$$

We'll find $f'(x)$,

$$f'(x) = \frac{d}{dx}\left[2(x-1)^{-\frac{1}{2}}\right]$$

$$f'(x) = -\frac{1}{2}(2)(x-1)^{-\frac{3}{2}}$$

$$f'(x) = -\frac{1}{\sqrt{(x-1)^3}}$$

and then evaluate $f'(x)$ at $a = 2$.

$$f'(2) = -\frac{1}{\sqrt{(2-1)^3}}$$

$$f'(2) = -\frac{1}{\sqrt{1}}$$

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$$f'(2) = -1$$

Therefore, using our linear approximation formula, we find that

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 2 + (-1)(x - 2)$$

$$L(x) = 2 - (x - 2)$$

$$L(x) = 2 - x + 2$$

$$L(x) = 4 - x$$

Now that we've built our linear approximation equation, we can plug in 1.9 to find an approximation of the value at that point.

$$L(1.9) = 4 - 1.9$$

$$L(1.9) = 2.1$$

The actual value of the function at 1.9 is

$$f(x) = \frac{2}{\sqrt{x - 1}}$$

$$f(1.9) = \frac{2}{\sqrt{1.9 - 1}}$$

$$f(1.9) = 2.1081$$

Therefore, you can see that the approximation $L(1.9)$, and the actual value $f(1.9)$, differ by

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$$f(1.9) - L(1.9) = 2.1081 - 2.1$$

$$f(1.9) - L(1.9) = 0.0081$$

Expressed as a percentage of $f(1.9) = 2.1081$, the error is only

$$\frac{0.0081}{2.1081} \cdot 100\,\% = 0.384\,\%$$

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